· Taylor's theorem
· Hessian matrix

Thm (Second derivative test) $f: \Omega (\subseteq \mathbb{R}^2) \to \mathbb{R}$ C^2-function.
$a \in \Omega$, $\nabla f(a) = 0$.

① $f_{xx} f_{yy} - f_{xy}^2 > 0$, $f_{xx} > 0$ at a $\Rightarrow$ a is a local min.
② " , $f_{xx} < 0$ " $\Rightarrow$ a " local max.
③ $f_{xx} f_{yy} - f_{xy}^2 < 0$ $\Rightarrow$ a is a saddle point.
④ $f_{xx} f_{yy} - f_{xy}^2 = 0$ at a $\Rightarrow$ inconclusive

Rmk ④; a can be local max, local min, saddle point.

eg1 $f(x,y) = 3x^2 - 10xy + 3y^2 + 2x + 2y + 3$.
Find and classify critical points of f.

(sol) f is a polynomial, it is differentiable on $\mathbb{R}^2$ C^2.

$$\nabla f = (f_x, f_y) = (6x - 10y + 2, \ -10x + 6y + 2)$$

$$\nabla f = 0 \Leftrightarrow \begin{cases} 6x - 10y + 2 = 0 \\ -10x + 6y + 2 = 0 \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{2} \\ y = \frac{1}{2} \end{cases}$$

$\left(\frac{1}{2}, \frac{1}{2}\right)$ is the only critical point of f.

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 6 & -10 \\ -10 & 6 \end{pmatrix}$$

$f_{xx}f_{yy} - f_{xy}^2 = 36 - 100 = -64 < 0$

By second derivative test, $(\frac{1}{2}, \frac{1}{2})$ is a saddle point. ☺

eg2 $f(x,y) = 3x - x^3 - 3xy^2$. Find all critical points and classify them.

(sol) f is a polynomial, hence differentiable on $\mathbb{R}^2$.

$\nabla f = (3 - 3x^2 - 3y^2, -6xy)$

$$\nabla f = 0 \Leftrightarrow \begin{cases} 3 - 3x^2 - 3y^2 = 0 & -① \\ -6xy = 0 & -② \end{cases}$$

② $\Rightarrow$ $x = 0$ or $y = 0$

$x = 0 \overset{①}{\Rightarrow} 3 - 3y^2 = 0 \Rightarrow y = \pm 1$

$y = 0 \Rightarrow 3 - 3x^2 = 0 \Rightarrow x = \pm 1$

$\therefore$ 4 critical points $(\pm 1, 0)$, $(0, \pm 1)$

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} -6x & -6y \\ -6y & -6x \end{pmatrix}$$

critical points	$Hf(a)$	$\det Hf(a) = f_{xx}f_{yy} - f_{xy}^2$	$f_{xx}(a)$	Nature of a
$(1,0)$	$\begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}$	$36 > 0$	$-6 < 0$	local max
$(-1,0)$	$\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$	$36 > 0$	$6 > 0$	local min
$(0,1)$	$\begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix}$	$-36 < 0$	no need to check	saddle point
$(0,-1)$	$\begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix}$	$-36 < 0$	"	saddle point

eg3 (Inconclusive cases from 2nd derivative test)
case ③ in the theorem.

$f(x,y) = x^2 + y^4$ $\quad$ $g(x,y) = x^2 - y^4$ $\quad$ $h(x,y) = -x^2 - y^4$

$\nabla f = (2x, 4y^3)$ $\quad$ $\nabla g = (2x, -4y^3)$ $\quad$ $\nabla h = (-2x, -4y^3)$

$\Rightarrow$ $(0,0)$ is the only critical point of f, g, h.

$Hf = \begin{pmatrix} 2 & 0 \\ 0 & 12y^2 \end{pmatrix}$ $\quad$ $Hg = \begin{pmatrix} 2 & 0 \\ 0 & -12y^2 \end{pmatrix}$ $\quad$ $Hh = \begin{bmatrix} -2 & 0 \\ 0 & -12y^2 \end{bmatrix}$

$Hf(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad$ $Hg(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad$ $Hh(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow$ All Hessian matrices have determinant 0 at $(0,0)$.

$\therefore$ 2nd derivative test is inconclusive.

However, $f(x,y) = x^2+y^4 \geq 0 = f(0,0)$

$\therefore f$ has local min at $(0,0)$.

$g(x,y) = x^2 - y^4$

$g(0,y) = -y^4 \leq 0$

$g(x,0) = x^2 \geq 0$

g has a saddle point at $(0,0)$.

$h(x,y) = -x^2 - y^4 \leq 0 = h(0,0)$

$\therefore h$ has local max at $(0,0)$.

2nd derivative test for general n.

Let $f: \Omega (\subseteq \mathbb{R}^n) \to \mathbb{R}$. C^2-function.

$a \in \Omega$, $\nabla f(a) = 0$.

$$Hf = \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ \vdots & & & \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{pmatrix}$$

f is $C^2 \Rightarrow Hf(a)$ is symmetric.

A fact from linear algebra: $\exists$ orthogonal matrix P

$\left(\text{i.e. } P \text{ satisfies } PP^T = P^TP = I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right)$

s.t. $$P^T \, Hf(a) \, P = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

where λ_i are eigenvalues of $Hf(a)$.

From this fact, we have

Thm $Hf(a)$ is $\begin{cases} \text{positive definite} & \Longleftrightarrow \text{All } \lambda_i > 0 \\ \text{negative definite} & \Longleftrightarrow \text{All } \lambda_i < 0 \\ \text{indefinite} & \Longleftrightarrow \text{some } \lambda_i > 0 \text{ and some } \lambda_j < 0 \end{cases}$

Another way to check definiteness of $Hf(a)$ $(k \leq n)$

Let H_k be the $k \times k$ submatrix of $Hf(a)$

: $$H_k = \begin{pmatrix} f_{x_1x_1} & \cdots & f_{x_1x_k} \\ & & \\ f_{x_kx_1} & \cdots & f_{x_kx_k} \end{pmatrix} \qquad Hf = \begin{pmatrix} \boxed{H_k} & \\ & \end{pmatrix}$$

① $Hf(a)$ is positive definite

$\iff \det H_k > 0$ for $k = 1, \cdots, n$

② $Hf(a)$ is negative definite

$\iff \det H_k \begin{cases} < 0 & \text{if } k \text{ is odd.} \\ > 0 & \text{if } k \text{ is even.} \end{cases}$

Rmk If $n=2$, $\det H_1 = \det(f_{xx}) = f_{xx}$

$\det H_2 = \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = f_{xx}f_{yy} - f_{xy}^2$

Same to the therem for 2 variables.

<u>Lagrange multipliers</u> : Find extrema under constraints.

eg1 Find the point $\underbrace{\text{on the parabola } x^2 = 4y}_{\text{Constraint}}$

$\underbrace{\text{closest}}_{\text{extrema}}$ to $(1, 2)$.

i.e. Find minimum of $f(x,y) = (x-1)^2 + (y-2)^2$

under constraint $\underbrace{g(x,y) = x^2 - 4y = 0}_{\text{expressed as a level set } g=0.}$

Thm (Lagrange multipliers)

Let f, g be C^1-functions on $\Omega (\subseteq \mathbb{R}^n)$.

$$S = g^{-1}(c) = \{x \in \Omega \mid g(x) = c\}$$

Suppose ① a is a local extremum of f on S

② $\nabla g(a) \neq 0$.

Then $\begin{cases} \nabla f(a) = \lambda \nabla g(a) \\ g(a) = c \end{cases}$ for some $\lambda \in \mathbb{R}$.

Rmk ① λ is called Lagrange multiplier.

② Let $F(\underset{x_1 \cdots x_n}{x}, \lambda) = f(x) - \lambda(g(x) - c)$.

Then $\nabla F(x, \lambda) = (\nabla(f_{(x)} - \lambda g_{(x)}), \; g(x) - c)$

Finding critical point of f under constraint $g = c$

$\Leftrightarrow$ Find critical point of F without constraint.

Back to eg $f(x, y) = (x-1)^2 + (y-2)^2$

$g(x, y) = x^2 - 4y$

minimize $f(x, y)$ under constraint $g(x, y) = 0$

f, g are differentiable on $\mathbb{R}^2$

$\nabla f = (2(x-1), 2(y-2))$

$\nabla g = (2x, -4) \neq 0$ on $\mathbb{R}^2$.

Suppose (x, y) is a local extremum of $f(x,y)$ on $g(x,y)=0$, then Lagrange multiplier method

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) & \text{for some } \lambda \in \mathbb{R}. \\ g(x,y) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} (2(x-1), 2(y-2)) = \lambda (2x, -4) \\ g(x,y) = x^2 - 4y = 0 \end{cases}$$

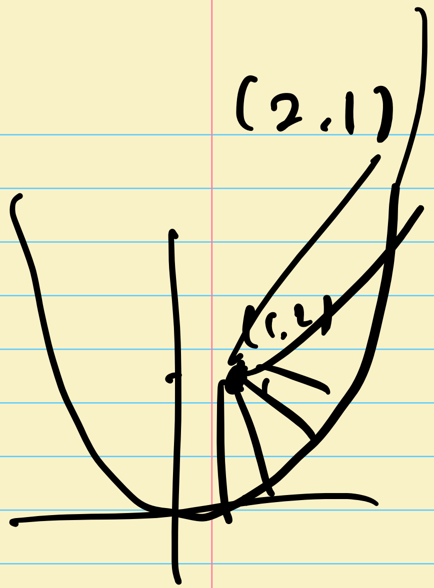
$$\Rightarrow \begin{cases} 2(x-1) = 2\lambda x & \Rightarrow x - 1 = \lambda x & \text{i.e. } x(1-\lambda) = 1 \\ 2(y-2) = -4\lambda & y - 2 = -2\lambda & y = 2(1-\lambda) = \frac{2}{x} \\ x^2 - 4y = 0 \end{cases}$$

$$x^2 - \frac{8}{x} = 0$$

$$x^3 = 8$$

$$\therefore x = 2$$

$$y = 1$$



$(2,1)$ is the only possible local extrema.

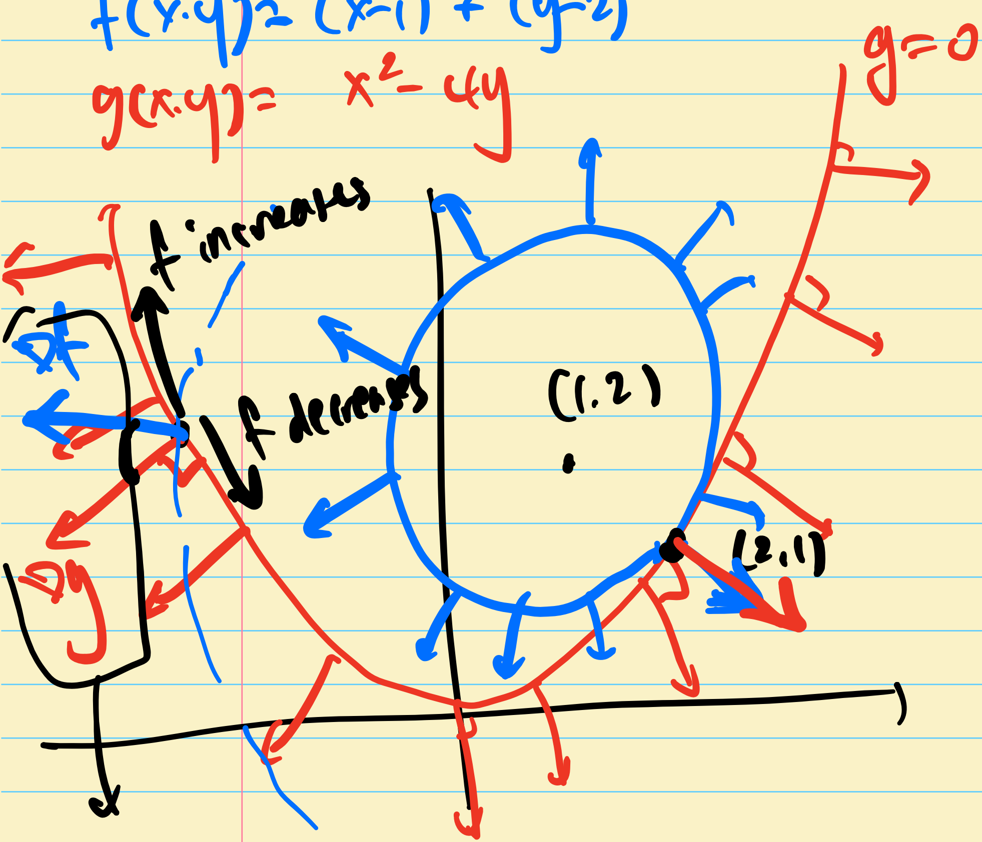
Geometrically, f must have a minimum on $g=0$.

$\Rightarrow$ f has minimum at $(2,1)$ on $g=0$.

$f(2,1) = (2-1)^2 + (1-2)^2 = 2$.

$f(x,y) = (x-1)^2 + (y-2)^2$

$g(x,y) = x^2 - 4y$



not parallel $\Rightarrow$ This point cannot be extremum.

Ex Find the point on the parabola $x^2=4y$ s.t. closest to $(2,5)$

$f(x,y) = (x-2)^2+(y-5)^2$

$g(x,y) = x^2-4y$

$\Rightarrow \begin{cases} \nabla f = \lambda \nabla g \\ g=0 \end{cases}$ has solutions $(4,4)$ $(-2,1)$

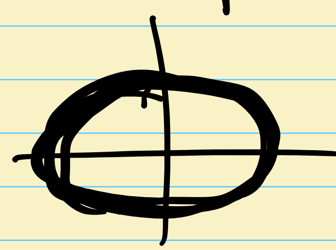
$(4,4)$: global min of f on $g=0$

$(-2,1)$: not local extrema on $g=0$.

eg2 Maximize xy^2 on the ellipse $x^2+4y^2=4$.

(sol) Let $f(x,y)=xy^2$, $g(x,y)=x^2+4y^2$

Note f is continuous and the ellipse $g^{-1}(4)$ is closed & bounded.



$\therefore$ By EVT, f has global max and min on $g=4$.

$\nabla f = (y^2, 2xy)$. $\nabla g = (2x, 8y)$

Note that $\nabla g \neq 0$ on $x^2+4y^2=4$.

Lagrange multiplier: $\begin{cases} \nabla f = \lambda \nabla g \\ g = 4 \end{cases} \iff \begin{cases} y^2 = 2\lambda x & \text{①} \\ 2xy = 8\lambda y & \text{②} \\ x^2 + 4y^2 = 4 & \text{③} \end{cases}$

If $y \neq 0$, ② $\Rightarrow 2x = 8\lambda \overset{①}{\Rightarrow} y^2 = 8\lambda^2$

$x = 4\lambda$

$\overset{③}{\Rightarrow} 16\lambda^2 + 32\lambda^2 = 4$

$\Rightarrow \lambda = \pm\sqrt{\frac{1}{12}} = \pm\frac{1}{2\sqrt{3}}$

$\Rightarrow x = \pm\sqrt{\frac{4}{3}}, \quad y = \pm\sqrt{\frac{2}{3}}.$

If $y = 0$, ③ $\Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

Total 6 points for candidates for local extremum found using Lagrange multipliers.

Compare values of $f(x,y) = xy^2$ at these points.

$f(\pm 2, 0) = 0$

$f\left(\sqrt{\frac{4}{3}}, \pm\sqrt{\frac{2}{3}}\right) = \sqrt{\frac{4}{3}} \cdot \frac{2}{3} = \frac{4}{3\sqrt{3}} \leftarrow \text{max}$

$$f\left(-\sqrt{\tfrac{4}{3}}, \pm\sqrt{\tfrac{2}{3}}\right) = -\sqrt{\tfrac{4}{3}} \cdot \tfrac{2}{3} = -\frac{4}{3\sqrt{3}} \leftarrow \text{min.}$$

$\therefore$ For $f(x,y)$ on $g(x,y)=4$,

global max value $= \dfrac{4}{3\sqrt{3}}$ at $\left(\sqrt{\tfrac{4}{3}}, \pm\sqrt{\tfrac{2}{3}}\right)$

" min " $= -\dfrac{4}{3\sqrt{3}}$ at $\left(-\sqrt{\tfrac{4}{3}}, \pm\sqrt{\tfrac{2}{3}}\right)$

$f : A \to \mathbb{R}$ $\quad$ $\text{int}(A)$

∂A

For problems of finding max/min of $f : A \to \mathbb{R}$, Lagrange multipliers can be used to study f on ∂A.

Eg Find global max/min of $f(x,y) = x^2 + 2y^2 - x + 3$ for $x^2 + y^2 \leq 1$.

(sol) f is continuous, $\{x^2+y^2 \leq 1\}$ closed and bounded.

By EVT, global max/min exist.

$$\begin{cases} a \in \text{int}(A) \Rightarrow \nabla f(a) = 0 \Rightarrow \text{only critical point } \left(\tfrac{1}{2}, 0\right),\ f\left(\tfrac{1}{2}, 0\right) = \tfrac{11}{4} \\ a \in \partial A = \{x^2 + y^2 = 1\} \end{cases}$$

we studied f on ∂A directly.

$(\cos\theta, \sin\theta) \in \partial A$

$f(\ \cdot\ ,\ \cdot\ ) = \cos^2\theta + 2\sin^2\theta - \cos\theta + 3$

We may use Lagrange multiplier to study f on $\partial A = g^{-1}(1)$, $g(x,y) = x^2+y^2$.

$Df = (2x-1, 4y)$

$Dg = (2x, 2y) \neq 0$ on ∂A.

$$\begin{cases} Df = \lambda Dg \\ g = 1 \end{cases} \Leftrightarrow \begin{cases} 2x-1 = 2\lambda x & \text{--- ①} \\ 4y = 2\lambda y & \text{--- ②} \\ x^2+y^2 = 1 & \text{--- ③} \end{cases}$$

② $\Rightarrow (4-2\lambda)y = 0 \Rightarrow \lambda = 2$ or $y = 0$.

If $\lambda = 2$, ① $\Rightarrow 2x-1 = 4x \Rightarrow x = -\frac{1}{2} \overset{③}{\Rightarrow} y = \pm\frac{\sqrt{3}}{2}$

If $y = 0$, ③ $\Rightarrow x = \pm 1$.

Comparing the values of f at these points

int A) $f(\frac{1}{2}, 0) = \frac{11}{4}$

∂A) $f(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}) = \frac{21}{4}$, $f(1,0) = 3$, $f(-1,0) = 5$

$\therefore$ Max $= \frac{21}{4}$ at $(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$

min $= \frac{11}{4}$ at $(\frac{1}{2}, 0)$.

If the level set $S=\{g=c\}$ is closed & bounded and f is continuous on S, by EVT, f has global extrema on S.

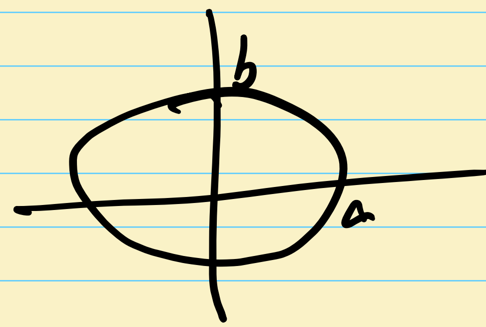
Quadratic constraints for 2-variables (conic section)

$$g(x,y)=Ax^2+Bxy+Cy^2+Dx+Ey+F$$

Some typical examples of $g=c$:

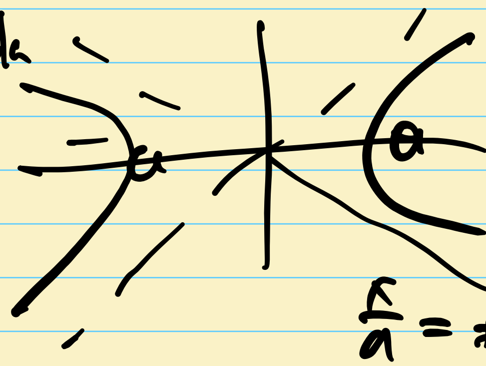
(i) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ $(a,b>0)$

ellipse



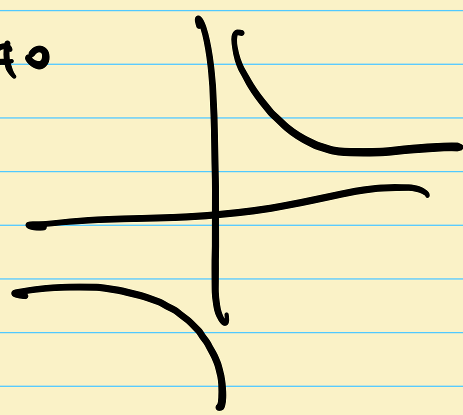
(ii) $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

hyperbola



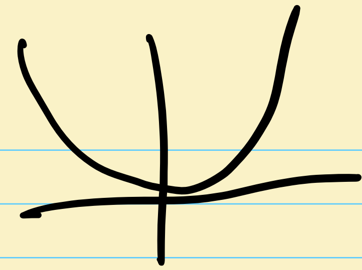
$\dfrac{x}{a}=\pm\dfrac{y}{b}$

(Rmk $xy=c$, $c\neq 0$



also a hyperbola)

(iii) $y = ax^2$, $a \neq 0$ parabola



(iv) degenerate cases

- $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$ → a point $(0,0)$
- " $= -1$ → empty set
- $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ → a pair of lines intersect
- $x^2 = c$ → $x = \pm\sqrt{c}$ ($c \geq 0$) a pair of lines parallel

<u>Fact</u> By a change of coordinates, any quadratic constraint $g(x,y) = c$ can be transformed to one of the forms above

(⇒ ellipse, hyperbola, parabola, degenerate case)